

# On lattice coverings of Nil space by congruent geodesic balls <sup>\*</sup>

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## Abstract

The Nil geometry, which is one of the eight 3-dimensional Thurston geometries, can be derived from W. Heisenberg's famous real matrix group.

The aim of this paper to study *lattice coverings* in Nil space. We introduce the notion of the density of considered coverings and give upper and lower estimations to it, moreover we formulate a conjecture for the ball arrangement of the least dense lattice-like geodesic ball covering and give its covering density  $\Delta \approx 1.42900615$ .

The homogeneous 3-spaces have a unified interpretation in the projective 3-sphere and in our work we will use this projective model of the Nil geometry.

## 1 Notions of the Nil geometry

In this Section we summarize the significant notions and denotations of the Nil geometry (see [1], [6]).

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The Nil geometry is a homogeneous 3-space derived from the famous real matrix group  $\mathbf{L}(\mathbb{R})$  discovered by Werner Heisenberg. The Lie theory with the method of the projective geometry makes possible to investigate and to describe this topic.

The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \quad (1.1)$$

defines "translations"  $\mathbf{L}(\mathbb{R}) = \{(x, y, z) : x, y, z \in \mathbb{R}\}$  on the points of the space  $\mathbf{Nil} = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ . These translations are not commutative in general. The matrices  $\mathbf{K}(z) \triangleleft \mathbf{L}$  of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (0, 0, z) \quad (1.2)$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of  $\mathbf{L}$ . The elements of  $\mathbf{K}$  are called *fibre translations*. Nil geometry of the Heisenberg group can be projectively (affinely) interpreted by the "right translations" on points as the matrix formula

$$(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \quad (1.3)$$

shows (see (1.1)). Here we consider  $\mathbf{L}$  as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex  $E_0(\mathbf{e}_0)E_1^\infty(\mathbf{e}_1)E_2^\infty(\mathbf{e}_2)E_3^\infty(\mathbf{e}_3)$ ,  $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$  with the unit point  $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ ) which is distinguished by an origin  $E_0$  and by the ideal points of coordinate axes, respectively. Moreover,  $\mathbf{y} = c\mathbf{x}$  with  $0 < c \in \mathbb{R}$  (or  $c \in \mathbb{R} \setminus \{0\}$ ) defines a point  $(\mathbf{x}) = (\mathbf{y})$  of the projective 3-sphere  $\mathcal{PS}^3$  (or that of the projective space  $\mathcal{P}^3$  where opposite rays  $(\mathbf{x})$  and  $(-\mathbf{x})$  are identified). The dual system  $\{(\mathbf{e}^i)\}$ ,  $(\{\mathbf{e}^i\} \subset \mathbf{V}_4)$  describes the simplex planes, especially the plane at infinity  $(\mathbf{e}^0) = E_1^\infty E_2^\infty E_3^\infty$ , and generally,  $\mathbf{v} = \mathbf{u}_c^1$  defines a plane  $(\mathbf{u}) = (\mathbf{v})$  of  $\mathcal{PS}^3$  (or that of  $\mathcal{P}^3$ ). Thus  $0 = \mathbf{x}\mathbf{u} = \mathbf{y}\mathbf{v}$  defines the incidence of point  $(\mathbf{x}) = (\mathbf{y})$  and plane  $(\mathbf{u}) = (\mathbf{v})$ , as  $(\mathbf{x})\mathbf{I}(\mathbf{u})$  also denotes it. Thus Nil can be visualized in the affine 3-space  $\mathbf{A}^3$  (so in  $\mathbf{E}^3$ ) as well.

The translation group  $\mathbf{L}$  defined by formula (1.3) can be extended to a larger group  $\mathbf{G}$  of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of Nil.

In [2] E. Molnár has shown that a rotation through angle  $\omega$  about the  $z$ -axis at the origin, as isometry of Nil, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in  $x, y$  to  $z$ -image  $\bar{z}$  as follows:

$$\begin{aligned}\mathcal{R} = \mathbf{r}(O, \omega) : (1; x, y, z) &\rightarrow (1; \bar{x}, \bar{y}, \bar{z}); \\ \bar{x} &= x \cos \omega - y \sin \omega, \quad \bar{y} = x \sin \omega + y \cos \omega, \\ \bar{z} &= z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2) \sin 2\omega + \frac{1}{2}xy \cos 2\omega.\end{aligned}\tag{1.4}$$

This rotation formula  $\mathcal{R}$ , however, is conjugate by the quadratic mapping  $\mathcal{M}$  to the linear rotation  $\Omega$  in (1.5) as follows

$$\begin{aligned}\mathcal{M} : (1; x, y, z) &\xrightarrow{\mathcal{M}} (1; x', y', z') = (1; x, y, z - \frac{1}{2}xy) \text{ to} \\ \Omega : (1; x', y', z') &\xrightarrow{\Omega} (1; x'', y'', z'') = (1; x', y', z') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \text{with } \mathcal{M}^{-1} : (1; x'', y'', z'') &\xrightarrow{\mathcal{M}^{-1}} (1; \bar{x}, \bar{y}, \bar{z}) = (1; x'', y'', z'' + \frac{1}{2}x''y'').\end{aligned}\tag{1.5}$$

This quadratic conjugacy modifies the Nil translations in (1.1), as well. Now a translation with  $(X, Y, Z)$  in (1.3) instead of  $(x, y, z)$  will be changed by the above conjugacy to the translation

$$(1; x, y, z) \longrightarrow (1; \bar{x}, \bar{y}, \bar{z}) = (1; x, y, z) \begin{pmatrix} 1 & X & Y & Z - \frac{1}{2}XY \\ 0 & 1 & 0 & -\frac{1}{2}Y \\ 0 & 0 & 1 & \frac{1}{2}Y \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.6}$$

that is again an affine collineation. We shall use the following important classification theorem.

**Theorem 1.1 (E. Molnár [2])** (1) *Any group of Nil isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (1.5) to an affine group of the affine (or Euclidean) space  $\mathbf{A}^3 = \mathbf{E}^3$  whose projection onto the  $(x, y)$  plane is an isometry group of  $\mathbf{E}^2$ . Such an affine group preserves a plane*

→ point polarity of signature  $(0, 0, \pm 0, +)$ .

(2) Of course, the involutive line reflection about the  $y$  axis

$$(1; x, y, z) \rightarrow (1; -x, y, -z),$$

preserving the Riemann metric, and its conjugates by the above isometries in (1) (those of the identity component) are also Nil-isometries. Orientation reversing Nil-isometry does not exist.

The geodesic curves of the Nil geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves  $g(x(t), y(t), z(t))$  in our model can be determined by the general theory of Riemann geometry: We can assume, that the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (1.1);

$$\begin{aligned} x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c \cos \alpha, \quad \dot{y}(0) = c \sin \alpha, \\ \dot{z}(0) = w; \quad -\pi \leq \alpha \leq \pi. \end{aligned}$$

The arc length parameter  $s$  is introduced by

$$s = \sqrt{c^2 + w^2} \cdot t, \text{ where } w = \sin \theta, \quad c = \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

i.e. unit velocity can be assumed.

**Remark 1.1** Thus we have harmonized the scales along the coordinate axes.

The equation systems of a helix-like geodesic curves  $g(x(t), y(t), z(t))$  if  $0 < |w| < 1$ :

$$\begin{aligned} x(t) &= \frac{2c}{w} \sin \frac{wt}{2} \cos \left( \frac{wt}{2} + \alpha \right), \quad y(t) = \frac{2c}{w} \sin \frac{wt}{2} \sin \left( \frac{wt}{2} + \alpha \right), \\ z(t) &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[ \left( 1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt} \right) + \right. \right. \\ &\quad \left. \left. + \left( 1 - \frac{\sin(2wt)}{wt} \right) - \left( 1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt} \right) \right] \right\} = \\ &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[ \left( 1 - \frac{\sin(wt)}{wt} \right) + \left( \frac{1 - \cos(2wt)}{wt} \right) \sin(wt + 2\alpha) \right] \right\}. \end{aligned} \quad (1.7)$$

In the cases  $w = 0$  the geodesic curve is the following:

$$x(t) = c \cdot t \cos \alpha, \quad y(t) = c \cdot t \sin \alpha, \quad z(t) = \frac{1}{2} c^2 \cdot t^2 \cos \alpha \sin \alpha. \quad (1.8)$$

The cases  $|w| = 1$  are trivial:  $(x, y) = (0, 0)$ ,  $z = w \cdot t$ .

**Definition 1.1** *The distance  $d(P_1, P_2)$  between the points  $P_1$  and  $P_2$  is defined by the arc length of the geodesic curve from  $P_1$  to  $P_2$ .*

## 1.1 On the geodesic ball

In our work [6] we have introduced the following definitions:

**Definition 1.2** *The geodesic sphere of radius  $R$  with centre at the point  $P_1$  is defined as the set of all points  $P_2$  in the space with the condition  $d(P_1, P_2) = R$ . Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in the Nil space.*

**Remark 1.2** *We will see that this last condition depends on radius  $R$ .*

**Definition 1.3** *The body of the geodesic sphere of centre  $P_1$  and of radius  $R$  in the Nil space is called geodesic ball, denoted by  $B_{P_1}(R)$ , i.e.  $Q \in B_{P_1}(R)$  iff  $0 \leq d(P_1, Q) \leq R$ .*

**Remark 1.3** *Henceforth, typically we choose the origin as centre of the sphere and its ball, by the homogeneity of Nil.*

We apply the quadratic mapping  $\mathcal{M} : \text{Nil} \longrightarrow \mathbf{A}^3$  at (1.5) to the geodesic sphere  $S$ , its  $\mathcal{M}$ -image is denoted by  $S' = \mathcal{M}(S)$ .

We choose a point  $P(x(R, \theta, \alpha), y(R, \theta, \alpha), z(R, \theta, \alpha))$  lying on a sphere  $S$  of radius  $R$  with centre at the origin. The coordinates of  $P$  are given by parameters  $(\alpha \in [-\pi, \pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], R > 0)$  (see (1.5), (1.10)), its  $\mathcal{M}$ -image is  $P'(x'(R, \theta, \alpha), y'(R, \theta, \alpha), z'(R, \theta, \alpha)) \in S'$  where

$$\begin{aligned} x'(R, \theta, \alpha) &= \frac{2c}{w} \sin \frac{wR}{2} \cos \left( \frac{wR}{2} + \alpha \right), \\ y'(R, \theta, \alpha) &= \frac{2c}{w} \sin \frac{wR}{2} \sin \left( \frac{wR}{2} + \alpha \right), \\ z'(R, \theta, \alpha) &= wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR, \quad (\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}), \\ \text{if } \theta &= 0 \text{ then } x'(R, 0, \alpha) = R \cos \alpha, \\ y'(R, 0, \alpha) &= R \sin \alpha, \quad z'(R, 0, \alpha) = 0. \end{aligned} \tag{1.9}$$

We can see from the last equations that  $(x')^2 + (y')^2 = \frac{4c^2}{w^2} \sin^2 \frac{wR}{2}$  and that the  $z'$ -coordinate does not depend on the parameter  $\alpha$ , therefore  $S'$  can be generated

by rotating the following curve about the  $z$  axis (lying in the plane  $[x, z]$ ):

$$\begin{aligned}
 X(R, \theta) &= \frac{2c}{w} \sin \frac{wR}{2} = \frac{2 \cos \theta}{\sin \theta} \sin \frac{R \sin \theta}{2}, \\
 Z(R, \theta) &= wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR = \\
 R \sin \theta + \frac{R \cos^2 \theta}{2 \sin \theta} - \frac{\cos^2 \theta}{2 \sin^2 \theta} \sin(R \sin \theta), \quad (\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}); \\
 \text{if } \theta = 0 \text{ then } X(R, 0) &= R, \quad Z(R, 0) = 0.
 \end{aligned} \tag{1.10}$$

**Remark 1.4** From the definition of the quadratic mapping  $\mathcal{M}$  at (1.5) it follows that the cross section of the spheres  $S$  and  $S'$  with the plane  $[x, z]$ , is the same curve which is specified by the parametric equations (1.10).

**Remark 1.5** The parametric equations of the geodesic sphere of radius  $R$  can be generated from (1.10) by Nil rotation (see (1.4)).

We have denoted by  $B(S)$  the body of the Nil sphere  $S$  and by  $B(S')$  the body of the sphere  $S'$ , furthermore we have denoted their volumes by  $Vol(B(S))$  and  $Vol(B(S'))$ , respectively.

In [6] we have proved the the following theorem:

**Theorem 1.2** The geodesic sphere and ball of radius  $R$  exists in the Nil space if and only if  $R \in [0, 2\pi]$ .

We obtain the volume of the geodesic ball of radius  $R$  by the following integral (see 1.10):

$$\begin{aligned}
 Vol(B(S)) &= 2\pi \int_0^{\frac{\pi}{2}} X^2 \frac{dZ}{d\theta} d\theta = \\
 &= 2\pi \int_0^{\frac{\pi}{2}} \left( \frac{2 \cos \theta}{\sin \theta} \sin \frac{(R \sin \theta)}{2} \right)^2 \cdot \left( -\frac{1}{2} \frac{R \cos^3 \theta}{\sin^2 \theta} + \frac{\cos \theta \sin(R \sin \theta)}{\sin \theta} + \right. \\
 &\quad \left. + \frac{\cos^3 \theta \sin(R \sin \theta)}{\sin^3 \theta} - \frac{1}{2} \frac{R \cos^3 \theta \cos(R \sin \theta)}{\sin^2 \theta} \right) d\theta.
 \end{aligned} \tag{1.11}$$

The Nil sphere of radius  $R$  is generated by the Nil rotation about the axis  $z$  (see the equation system (1.10) and remarks (1.4), (1.5)). The parametric equation

system of the geodesic sphere  $S(R)$  in our model:

$$\begin{aligned}
x(R, \theta, \phi) &= \frac{2c}{w} \sin \frac{wR}{2} \cdot \cos \phi = \frac{2 \cos \theta}{\sin \theta} \sin \frac{R \sin \theta}{2} \cdot \cos \phi, \\
y(R, \theta, \phi) &= \frac{2c}{w} \sin \frac{wR}{2} \cdot \sin \phi = \frac{2 \cos \theta}{\sin \theta} \sin \frac{R \sin \theta}{2} \cdot \sin \phi, \\
z(R, \theta, \phi) &= wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR + \frac{1}{4} \left( \frac{2c}{w} \sin \frac{wR}{2} \right)^2 \sin 2\phi = \\
&= R \sin \theta + \frac{R \cos^2 \theta}{2 \sin \theta} - \frac{\cos^2 \theta}{2 \sin^2 \theta} \sin(R \sin \theta) + \frac{1}{4} \left( \frac{2 \cos \theta}{\sin \theta} \sin R \frac{\sin \theta}{2} \right)^2 \sin 2\phi \\
&\quad -\pi < \phi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } \theta \neq 0. \\
\text{if } \theta = 0 \text{ then } x(R, 0, \phi) &= R \cos \phi, \quad y(R, 0, \phi) = R \sin \phi, \\
z(R, 0, \phi) &= \frac{1}{2} R^2 \cos \phi \sin \phi.
\end{aligned} \tag{1.12}$$

The following theorem was obtained by the derivatives of these parametrically represented functions (by intensive and careful computations with *Maple* through the second fundamental form) (see [6]):

**Theorem 1.3** *The geodesic Nil ball  $B(S(R))$  is convex in affine-Euclidean sense in our model if and only if  $R \in [0, \frac{\pi}{2}]$ .*

## 1.2 The discrete translation group $L(\mathbb{Z}, k)$

We consider the Nil translations defined in (1.1) and (1.3) and choose two arbitrary translations

$$\tau_1 = \begin{pmatrix} 1 & t_1^1 & t_1^2 & t_1^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_1^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tau_2 = \begin{pmatrix} 1 & t_2^1 & t_2^2 & t_2^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.13}$$

now with upper indices for coordinate variables. We define the translation  $(\tau_3)^k$ , ( $k \in \mathbb{N}$ ,  $k \geq 1$ ) by the following commutator:

$$(\tau_3)^k = \tau_2^{-1} \tau_1^{-1} \tau_2 \tau_1 = \begin{pmatrix} 1 & 0 & 0 & -t_2^1 t_1^2 + t_1^1 t_2^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.14}$$

If we take integers as coefficients, their set is denoted by  $\mathbb{Z}$ , then we will generate the discrete group  $(\langle \tau_1, \tau_2 \rangle, k)$  denoted by  $\mathbf{L}(\tau_1, \tau_2, k)$  or by  $\mathbf{L}(\mathbb{Z}, k)$ .

We know that the orbit space  $\text{Nil}/\mathbf{L}(\mathbb{Z}, k)$  is a compact manifold, i.e. a Nil space form.

**Definition 1.4** The Nil point lattice  $\Gamma_P(\tau_1, \tau_2, k)$  is a discrete orbit of point  $P$  in the Nil space under the group  $\mathbf{L}(\tau_1, \tau_2, k) = \mathbf{L}(\mathbb{Z}, k)$  with an arbitrary starting point  $P$  for all  $(k \in \mathbb{N}, k \geq 1)$ .

**Remark 1.6** For simplicity, we have chosen the origin as starting point, by the homogeneity of Nil.

**Remark 1.7** We can assume that  $t_1^2 = 0$ , i.e. the image of the origin by the translation  $\tau_1$  lies on the plane  $[x, z]$ .

In the following we investigate the most important case  $k = 1$  where  $\tau_3$  correspond to the fibre translation, i.e.  $\tau_3 = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1$ .

We illustrate the action of  $\mathbf{L}(\mathbb{Z}, 1)$  on the Nil space in Fig. 1. We consider a non-convex polyhedron  $\mathcal{F} = OT_1T_2T_3T_{12}T_{21}T_{23}T_{213}T_{13}$ , in Euclidean sense, which is determined by translations  $\tau_1, \tau_2, \tau_3$ . This polyhedron determines a solid  $\tilde{\mathcal{F}}$  in the Nil space whose images under  $\mathbf{L}(\mathbb{Z}, 1)$  fill the Nil space just once, i.e. without gap and overlap.

Analogously to the Euclidean integer lattice and parallelepiped, the solid  $\tilde{\mathcal{F}}$  can be called Nil parallelepiped.

$\tilde{\mathcal{F}}$  is a fundamental domain of  $\mathbf{L}(\mathbb{Z}, 1)$ . The homogeneous coordinates of the vertices of  $\tilde{\mathcal{F}}$  can be determined in our affine model by the translations (1.13) and (1.14) with the parameters  $t_i^j$ ,  $k = 1$  ( $i \in \{1, 2\}$ ,  $j \in \{1, 2, 3\}$ ) (see Fig. 1 and (1.15)).

$$\begin{aligned}
& T_1(1, t_1^1, 0, t_1^3), T_2(1, t_2^1, t_2^2, t_2^3), T_3(1, 0, 0, \frac{t_1^1 t_2^2}{k}), \\
& T_{13}(1, t_1^1, 0, \frac{t_1^1 t_2^2}{k} + t_1^3), T_{12}(1, t_1^1 + t_2^1, t_2^2, t_2^3 + t_1^3), \\
& T_{21}(1, t_1^1 + t_2^1, t_2^2, t_1^1 t_2^2 + t_1^3 + t_2^3), T_{23}(1, t_2^1, t_2^2, t_2^3 + \frac{t_1^1 t_2^2}{k}), \\
& T_{213} = T_{231}(1, t_1^1 + t_2^1, t_2^2, (k+1)\frac{t_1^1 t_2^2}{k} + t_1^3 + t_2^3).
\end{aligned} \tag{1.15}$$



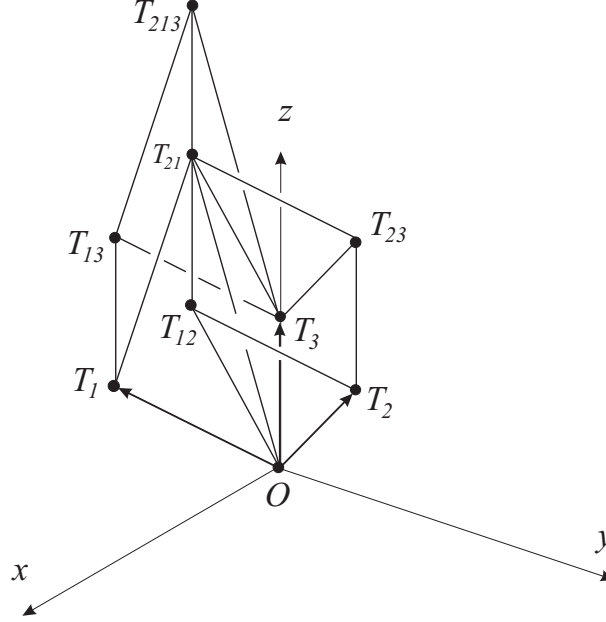


Figure 1:

We have determined in [6] the volume of  $\tilde{\mathcal{F}}$  by the following integral:

$$\text{Vol}(\tilde{\mathcal{F}}) = \int_0^{t_2^2} \int_0^{t_1^1} |t_1^1 \cdot t_2^2| \, dx dy = (t_1^1 \cdot t_2^2)^2. \quad (1.16)$$

From this formula it can be seen that the volume of the Nil parallelepiped depends on two parameters, i.e. on its projection onto the  $[x, y]$  plane.

## 2 The lattice-like geodesic ball coverings

A family of subsets  $\mathcal{K} = (K_i)_{i \in I}$  of Nil,  $I$  is a set of indices is called *covering* of Nil if each point of Nil belongs at least one of the set  $\{K_i, i \in I\}$ , i.e.  $\text{Nil} = \bigcup_{i \in I} K_i$ . A covering of Nil space is a *lattice covering* if it is of the form  $\mathcal{K} = (K_0 + \mathbf{v})_{\mathbf{v} \in \mathbf{L}(\mathbb{Z}, k)}$  where  $K_0$  is a element of  $\{K_i, i \in I\}$  i.e. the lattice coverings those coverings which cover Nil by translated copies of a single body  $K_0$  and in addition the translates are vectors of a lattice  $\mathbf{L}(\mathbb{Z})$ .

In following, we are only considering lattice coverings consisting of geodesic balls of the Nil. Let  $\mathcal{B}_{\Gamma}^c(R)$  denote a geodesic ball covering of Nil space with balls

$B^c(R)$  of radius  $R$  where their centres give rise to a Nil point lattice  $\Gamma(\tau_1, \tau_2, 1)$ .  $\tilde{\mathcal{F}}_0$  is an arbitrary Nil parallelepiped of this lattice (see (1.13), (1.14)). The images of  $\tilde{\mathcal{F}}_0$  by our discrete translation group  $\mathbf{L}(\tau_1, \tau_2, 1)$  cover the Nil space without overlap.

**Remark 2.1** *In the  $d$ -dimensional Euclidean space  $\mathbf{E}^d$ , ( $d \geq 1$ ) an arbitrary lattice  $\Gamma(\tau_1, \tau_2, 1)$  under the group  $\mathbf{L}(\mathbb{Z}, 1)$  gives a lattice covering of equal balls if the radius  $R$  of the balls is large enough, but this is not true in the Nil space, because of a geodesic ball exists in the Nil space if and only if  $R \in [0, 2\pi]$  (see Theorem 1.2).*

If we start with a lattice covering  $\mathcal{B}_\Gamma^c(R)$  and shrink the balls until they finally do not cover the space any more, then the threshold value of the shrinking radius  $R$  defines the least dense covering of equal balls to a given lattice  $\Gamma(\tau_1, \tau_2, 1)$ . The threshold value  $R^c$  is called *covering radius* of the point lattice  $\Gamma(\tau_1, \tau_2, 1)$ :

$$R^c := \min\{R : \text{where } \mathcal{B}_\Gamma^c(R) \text{ lattice covering by } \Gamma(\tau_1, \tau_2, 1)\}. \quad (2.1)$$

For the density of the packing it is sufficient to relate the volume of the "minimal covering ball" to that of the solid  $\tilde{\mathcal{F}}_0$ .

Analogously to the Euclidean case it can be defined the density  $\delta(R^c, \tau_1, \tau_2, 1)$  of the lattice-like geodesic ball covering  $\mathcal{B}_\Gamma(R^c)$ :

**Definition 2.2**

$$\Delta(R^c, \tau_1, \tau_2, 1) := \frac{\text{Vol}(\mathcal{B}_\Gamma(R^c))}{\text{Vol}(\tilde{\mathcal{F}})}, \quad (2.2)$$

*The main problem is that to which lattice  $\mathbf{L}(\tau_1, \tau_2, 1)$  belongs the minimal density  $\Delta_{opt}$ . We introduce for the "optimal arrangement" the following denotations:*

$$\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) := \min \left\{ \frac{\text{Vol}(\mathcal{B}_\Gamma(R^c))}{\text{Vol}(\tilde{\mathcal{F}})} \right\}. \quad (2.3)$$

**Remark 2.3** *The covering radius is the radius of the circumsphere of the lattice's Dirichlet-Voronoi polytope, that is the largest distance between the midpoint and the vertices of its Dirichlet-Voronoi polytope.*

## 2.1 The lattice-like ball covering of the lattice $L_{opt}(\tau_1^{opt}, \tau_2^{opt}, 1)$

First we consider the ball arrangement  $\mathcal{B}_\Gamma^{opt}(R_{opt})$  of the *densest lattice-like geodesic ball packing* in the Nil space, given by formulas (2.4), (2.5) (see [6]).

$$\begin{aligned} (a) \quad & d(O, T_1) = 2R = d(T_1, T_3), \\ (b) \quad & d(O, T_2) = 2R = d(T_2, T_3), \\ (c) \quad & d(T_1, T_2) = 2R, \\ (d) \quad & d(O, T_3) = 2R. \end{aligned} \tag{2.4}$$

$$\begin{aligned} t_1^{1,opt} &\approx 1.30633820, \quad t_1^{3,opt} = R_{opt}, \quad R_{opt} \approx 0.73894461; \\ t_2^{1,opt} &\approx 0.65316910, \quad t_2^{2,opt} \approx 1.13132206, \quad t_2^{3,opt} \approx 1.10841692, \\ T_1^{opt} &= (1, t_1^{1,opt}, 0, t_1^{3,opt}), \quad T_2^{opt} = (1, t_2^{1,opt}, t_2^{2,opt}, t_2^{3,opt}). \end{aligned} \tag{2.5}$$

This packing can be generated by the translations  $L_{opt}(\tau_1^{opt}, \tau_2^{opt}, 1)$  where  $\tau_1^{opt}$  and  $\tau_2^{opt}$  are given by the coordinates  $t_i^{j,opt}$   $i = 1, 2$ ;  $j = 1, 2, 3$  (see (2.5)). Thus we obtain the neighbouring balls around an arbitrary ball of the packing  $\mathcal{B}_\Gamma^{opt}(R_{opt})$  by the lattice  $\Gamma(\tau_1^{opt}, \tau_2^{opt}, 1)$ . We have ball "columns" in  $z$ -direction and in regular hexagonal projection onto the  $[x, y]$ -plane.

The Nil parallelepiped  $\tilde{\mathcal{F}} = OT_1^{opt}T_2^{opt}T_3^{opt}T_{12}^{opt}T_{21}^{opt}T_{23}^{opt}T_{213}^{opt}T_{13}^{opt}$  is a *fundamental domain* of  $L(\mathbb{Z}, 1)$ . The homogeneous coordinates of its vertices are known (see Fig. 1 and (1.15)).

We examine the *covering radius*  $R_p^c$  to the lattice  $\Gamma(\tau_1^{opt}, \tau_2^{opt}, 1)$ .

$$R_p^c := \min\{R : \text{where } \mathcal{B}_\Gamma^c(R) \text{ lattice covering by } \Gamma(\tau_1^{opt}, \tau_2^{opt}, 1)\}.$$

It is sufficient to investigate such ball arrangements  $\mathcal{B}_{\Gamma_{opt}}^c(R)$  where the balls cover  $\tilde{\mathcal{F}}$  or the Nil solid  $\tilde{\mathcal{P}} = OT_1^{opt}T_{12}^{opt}T_2^{opt}T_3^{opt}T_{13}^{opt}T_{21}^{opt}T_{23}^{opt}$  (see Section 1.2).

From (2.4) and (2.5) follows, that the point sets  $\{0, T_1^{opt}, T_2^{opt}, T_3^{opt}\}$ ,  $\{T_3^{opt}, T_1^{opt}, T_{23}^{opt}, T_2^{opt}\}$ ,  $\{T_3^{opt}, T_1^{opt}, T_{23}^{opt}, T_2^{opt}\}$ ,  $\{T_{12}^{opt}, T_1^{opt}, T_{23}^{opt}, T_2^{opt}\}$ ,  $\{T_{12}^{opt}, T_1^{opt}, T_{23}^{opt}, T_2^{opt}\}$ ,  $\{T_{12}^{opt}, T_1^{opt}, T_{23}^{opt}, T_2^{opt}\}$  are congruent by Nil isometries. The radius  $R$  of each circumscribed ball to the above point sets can be determined by the following system of equation:

$$d(O, C) = d(C, T_3^{opt}) = d(C, T_1^{opt}) = d(C, T_2^{opt}), \tag{2.6}$$

where  $C(1, c^1, c^2, c^3)$  is the center of the circumscribed ball of the point set  $\{0, T_1^{opt}, T_2^{opt}, T_3^{opt}\}$  ( $d$  is the Nil distance, see Definition 1.1):

$$\begin{aligned} c^1 &\approx 0.45981062, \quad c^2 \approx 0.26547179, \quad c^3 \approx 0.79997799, \\ R &\approx 0.90293941. \end{aligned} \tag{2.7}$$

**Remark 2.4**  $C(1, c^1, c^2, c^3)$  is a vertex of the Dirichlet-Voronoi domain of the point  $O$  in the Nil space.

$R \in [0, \frac{\pi}{2}]$  thus by Theorem 1.3 the ball of radius  $R$  is convex in affin-Euclidean sense. We form tetrahedra  $OT_1^{opt}T_2^{opt}T_3^{opt}$ ,  $T_3^{opt}T_1^{opt}T_2^{opt}T_{23}^{opt}$ ,  $T_3^{opt}T_1^{opt}T_{23}^{opt}T_2^{opt}$ ,  $T_{12}^{opt}T_1^{opt}T_{23}^{opt}T_2^{opt}$ ,  $T_{12}^{opt}T_1^{opt}T_{23}^{opt}T_{13}^{opt}$ ,  $T_{12}^{opt}T_{21}^{opt}T_{23}^{opt}T_{13}^{opt}$  in Euclidean sense, which fill the Nil solid  $\tilde{\mathcal{P}} = OT_1^{opt}T_{12}^{opt}T_2^{opt}T_3^{opt}T_{13}^{opt}T_{21}^{opt}T_{23}^{opt}$  just once. Their circumscribed congruent Nil balls are convex thus they cover the tetrahedra and so the ball arrangement  $\mathcal{B}_{\Gamma^{opt}}^c(R)$  cover the Nil solid  $\tilde{\mathcal{P}}$ . Thus the radius  $R$  of circumscribed ball give us the covering radius  $R_p^c$  to the lattice  $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt})$  and we get by (1.11), (1.16) and by the Definition 2.2 the following results:

$$\begin{aligned} Vol(B(R_p^c)) &\approx 3.12538516, \quad Vol(\tilde{\mathcal{P}}) = Vol(\tilde{\mathcal{F}}) \approx 2.18415656, \\ \Delta(R_p^c, \tau_1^{opt}, \tau_2^{opt}, 1) &:= \frac{Vol(\mathcal{B}_{\Gamma}^c(R_p^c))}{Vol(\tilde{\mathcal{F}})} \approx 1.43093459. \end{aligned} \quad (2.8)$$

It follows that

$$\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) \leq \Delta(R_p^c, \tau_1^{opt}, \tau_2^{opt}, 1) \approx 1.43093459. \quad (2.9)$$

**Remark 2.5** The density of the least dense lattice-like ball covering in the the Euclidean space is

$$\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) < \Delta_E = \frac{5\sqrt{5}\pi}{24} \approx 1.46350307.$$

## 2.2 Upper estimation for the covering radius

We consider a arbitrary lattice covering  $\mathcal{B}_{\Gamma}^c(R^c)$  where  $\Gamma = \Gamma(\tau_1, \tau_2, 1)$  and

$$R^c := \inf\{R : \text{where } \mathcal{B}_{\Gamma}^c(R) \text{ lattice covering by } \Gamma(\tau_1, \tau_2, 1)\}.$$

The *fundamental domain*  $\tilde{\mathcal{F}}$  of the translations group  $\mathbf{L}_{opt}(\tau_1, \tau_2, 1)$  and its vertices with their homogeneous coordinates can be seen in our affine model in (1.15) with parameters  $t_i^j$ ,  $i \in \{1, 2\}$ ,  $j \in \{1, 2, 3\}$  (see Fig. 1). We divide the Nil solid  $\tilde{\mathcal{P}} = OT_1T_{12}T_2T_3T_{13}T_{21}T_{23}$  (have been derived from the fundamental domain  $\tilde{\mathcal{F}}$ ,

see Section 1.2) into Nil solids  $OT_1T_2T_3$ ,  $T_3T_1T_{23}T_{13}$ ,  $T_3T_1T_{23}T_2$ ,  $T_{12}T_1T_{23}T_2$ ,  $T_{12}T_1T_{23}T_{13}$ ,  $T_{12}T_{21}T_{23}T_{13}$  which are tetrahedra in terms of Euclidean geometry. It is clear, that one of them contain the centre of its circumscribed Nil ball. Suppose now, that this "tetrahedron" is  $OT_1T_2T_3$  of which circumscribed ball  $B(R)$  centred by  $C$  passing through the points  $O, T_1, T_2, T_3$ . We note here that from conditions of the Nil ball follows that the ball  $B(R)$  contain the "Euclidean line segment"  $OT_3 = \sqrt{\text{Vol}(\widetilde{\mathcal{F}})}$  (see Section 1.1-2). From the Definition 2.2 follows, that

$$\Delta(R^c, \tau_1, \tau_2, 1) \geq \Delta(R, \tau_1, \tau_2, 1) = \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{\text{Vol}(\widetilde{\mathcal{F}})} = \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{OT_3^2}. \quad (2.10)$$

**Remark 2.6** A geodesic curve in Nil space "parallel to the axis  $z$ " correspond to an "Euclidean line segment" (see (1.5), (1.7)) with same lenght.

Forther estimation for the density we need to investigate the upper bound of the length of the line segment  $OT_3 = \sqrt{\text{Vol}(\widetilde{\mathcal{F}})}$  in  $B(R)$ .

Let  $S_*(R)$  be a geodesic sphere of radius  $R$  with centre at the origin. We apply the quadratic mapping  $\mathcal{M} : \text{Nil} \rightarrow \mathbf{A}^3$  at (1.5) to the geodesic sphere  $S_*$ , its  $\mathcal{M}$ -image is denoted by  $S'_* = \mathcal{M}(S_*)$ , moreover we have denoted by  $B(S_*(R))$  the body of the Nil sphere  $S_*(R)$  and by  $B(S'_*(R))$  the body of the sphere  $S'_*(R)$ ,

**Lemma 2.7** The length of the vertical chords of  $S_*(R)$  do not change at  $\mathcal{M}$ .

The proof of this lemma follows from the definition of the quadratic mapping  $\mathcal{M}$ .

**Lemma 2.8**  $B(S'_*(R)) = \mathcal{M}(B(S_*(R)))$  is convex in our model in Euclidean sense if and only if  $R \in [0, \pi]$ .

**Proof:** From the Section 1.1 can be seen that  $S'_*(R)$  can be generated by rotating the curve  $\mathcal{C}(\theta) = (X(R, \theta), Z(R, \theta))$  ( $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ ) (see 1.10) about the  $z$  axis (lying in the plane  $[x, z]$ ) thus the convexity of the ball  $B(S'_*(R))$  follows from the investigation of the derivative

$$\begin{aligned} & \frac{dZ(R, \theta)}{d\theta} = \\ &= \cos(\theta) \left( \frac{(-(\cos(\theta))^2 R \sin(\theta) + 2 \sin(\sin(\theta) R) (\sin(\theta))^2)}{2 (\sin(\theta))^3} - \right. \\ & \left. - \frac{((\cos(\theta))^2 \cos(\sin(\theta) R) t \sin(\theta) + 2 (\cos(\theta))^2 \sin(\sin(\theta) R))}{2 (\sin(\theta))^3} \right). \end{aligned} \quad (2.11)$$

From the first (2.11) and second derivatives of  $Z(R, \theta)$  follows that if  $R \in [0, \pi]$  then the ball  $B(S'_*(R))$  convex. If  $R \in (\pi, 2\pi]$  then the equation  $\frac{dZ(R, \theta)}{d\theta} = 0$  possesses a solution and the curve  $\mathcal{C}$  has an inflection point in the interval  $\theta \in (0, \frac{\pi}{2})$ . In Fig. 2 can be seen the complete curve  $\mathcal{C}(\theta) = (X(R, \theta), Z(R, \theta))$  for the parameter  $R = 2\pi$ , then the curve possesses at the point  $\theta = \frac{\pi}{6}$  the maximum,  $Z(2\pi, \frac{\pi}{6}) = \frac{5\pi}{2}$ .

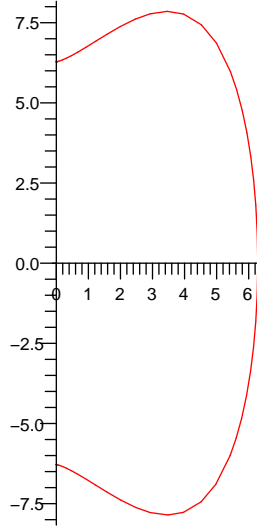


Figure 2:

**Lemma 2.9** *If  $R \in [\frac{\pi}{2}, 2\pi]$  then*

$$\begin{aligned} \Delta(R^c, \tau_1, \tau_2, 1) &\geq \Delta(R, \tau_1, \tau_2, 1) \geq \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{(2R)^2} > \\ &> \Delta(R_p^c, \tau_1^{\text{opt}}, \tau_2^{\text{opt}}, 1) \approx 1.43093459. \end{aligned} \quad (2.12)$$

**Proof:**

1.  $R \in [\frac{\pi}{2}, \pi]$

We get by (2.10) and by Lemmas 2.6-7 the following inequalities:

$$\Delta(R^c, \tau_1, \tau_2, 1) \geq \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{OT_3^2} \geq \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{(2R)^2}. \quad (2.13)$$

The function  $f(R) = \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{(2R)^2}$  depends only on the parameter  $R$  and its graph which is increasing on the interval  $R \in [\frac{\pi}{2}, \pi]$ , can be seen in

Fig. 3. Consequently, at the point  $R = \frac{\pi}{2}$  the function possesses a minimum,  $f(\frac{\pi}{2}) \approx 1.71179510 > \Delta(R_p^c, \tau_1^{opt}, \tau_2^{opt}, 1) \approx 1.43093459$ .

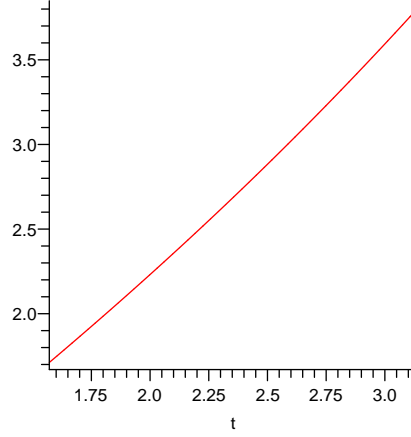


Figure 3:

2.  $R \in [\pi, 2\pi]$

We divide this interval into two part,  $[\pi, 2\pi] = [\pi, \frac{3\pi}{2}] \cup (\frac{3\pi}{2}, 2\pi]$ .

(a)  $R \in [\pi, \frac{3\pi}{2}]$

In this interval the greatest vertical chord  $h_1$  for the balls  $B(R)$  is  $h_1 = \frac{13\pi}{4} \approx 10.21017613$ .

$$\Delta(R^c, \tau_1, \tau_2, 1) \geq \frac{\text{Vol}(\mathcal{B}_\Gamma(R) \cap \tilde{\mathcal{F}})}{h_1^2} := f_1(R). \quad (2.14)$$

The Fig. 4.a show its increasing function on the interval  $R \in [\pi, \frac{3\pi}{2}]$ . It is clear that this function possesses a minimum, at the point  $R = \pi$ .  $f_1(\pi) \approx 1.441711246 > \Delta(R_p^c, \tau_1^{opt}, \tau_2^{opt}, 1) \approx 1.43093459$ .

(b)  $R \in (\frac{3\pi}{2}, 2\pi]$

Similarly to 2/a case we examine the possible greatest vertical chord  $h_2$  for the balls  $B(R)$  on the given interval. We get that  $h_2 = 5\pi \approx 15.70796327$  (see Fig. 5).

$$\Delta(R^c, \tau_1, \tau_2, 1) \geq \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{h_2^2} := f_2(R). \quad (2.15)$$

The Fig. 4.b shows the graph of the  $f_2(R)$  on the interval  $[\frac{3\pi}{2}, 2\pi]$ . It is evident that this function has a minimum, at the point  $R = \frac{3\pi}{2}$ .  $f_2(\frac{3\pi}{2}) \approx 2.372757787 > \Delta(R_p^c, \tau_1^{opt}, \tau_2^{opt}) \approx 1.43093459$ .

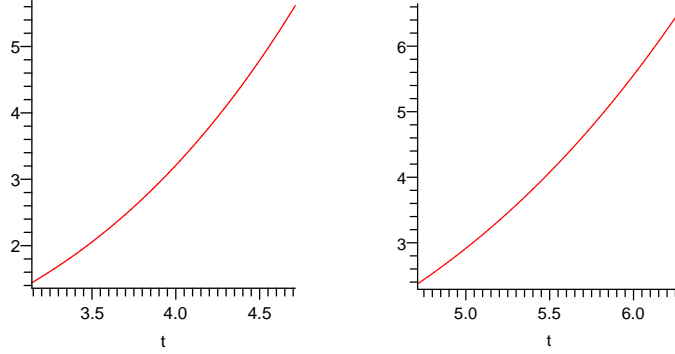


Figure 4: a, b

Immediate consequence of the Lemma 2.8 is the following Theorem:

**Theorem 2.10** *The radius  $R_{opt}^c$  which belongs to the minimal density  $\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1)$  of lattice-like geodesic ball coverings, is found in the interval  $[0, \frac{\pi}{2}]$ .*

**Remark 2.11** *The optimal covering Nil ball  $B(R_{opt}^c)$  is convex in Euclidean sense (see Theorem 1.3 and [6]).*

### 2.3 Lower bound to the covering density $\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1)$

In this section we consider a arbitrary lattice covering  $\mathcal{B}_\Gamma^c(R^c)$  where  $\Gamma = \Gamma(\tau_1, \tau_2, 1)$  (see (2.1-2)). The *fundamental domain*  $\tilde{\mathcal{F}}$  of the translations group  $\mathbf{L}(\tau_1, \tau_2, 1)$  is given by its vertices in our affine model in (1.15) (see Section 1.2 and Fig. 1). Similarly to Section 2.2 we divide the Nil solid  $\tilde{\mathcal{P}} = OT_1T_{12}T_2T_3T_{13}T_{21}T_{23}$  (have been derived from the fundamental domain  $\tilde{\mathcal{F}}$ , see Section 1.2) into Nil solids  $OT_1T_2T_3$ ,  $T_3T_1T_{23}T_{13}$ ,  $T_3T_1T_{23}T_2$ ,  $T_{12}T_1T_{23}T_2$ ,  $T_{12}T_1T_{23}T_{13}$ ,  $T_{12}T_{21}T_{23}T_{13}$  which are tetrahedra in Euclidean sense. It is clear, that one of them contain the centre of its circumscribed Nil ball and thus can be assumed, that this "tetrahedron" is  $OT_1T_2T_3$  of which circumscribed ball  $B(R)$  centred by  $C(1, c^1, c^2, c^3)$  and passing through the points  $O, T_1, T_2, T_3$ . By Theorem 2.10 can be supposed



that  $R \in [0, \frac{\pi}{2}]$  thus  $B(R)$  is a convex geodesic Nil ball in Euclidean sense and contain the Euclidean tetrahedron  $OT_1T_2T_3$ .

In order to find a lower bound to the covering density we investigate the density function

$$\Delta(R, \tau_1, \tau_2, 1) = \frac{Vol(\mathcal{B}_\Gamma(R))}{Vol(\tilde{\mathcal{F}})} = \frac{Vol(\mathcal{B}_\Gamma(R))}{OT_3^2} \quad (2.16)$$

to a given volume of parallelepiped  $Vol(\tilde{\mathcal{F}}) = OT_3^2$ . We have to find the minimum radius of the circumscribed ball of the Nil solid  $OT_1T_2T_3$  if the Euclidean line segment  $OT_3$  is given.

We project the points  $T_1$  and  $T_2$  parallel to the  $z$  axis onto the equidistance surface of  $O$  and  $T_3$  which is a hyperbolic paraboloid in our model with equation  $2z - xy = t_1^1 \cdot t_2^2$ . Their images are  $T_1^p$  and  $T_2^p$ .

**Lemma 2.12** *The radius  $R^p$  of the circumscribed ball with centre  $C^p(1, c_p^1, c_p^2, c_p^3)$  of the Nil solid  $OT_1^pT_2^pT_3$  which is a tetrahedron in terms of the Euclidean geometry, at most  $R$ .*

**Proof:** From the conditions of the Nil balls follows (see Section 1.1, [6] and Fig. 2) that vertical line segments from the point upto the equidistance surface are contained by the circumscribed ball  $B(R)$ , thus  $R^p \leq R$ . Moreover, during this projection the volume  $Vol(\tilde{\mathcal{F}}) = OT_3^2 = |t_1^1 \cdot t_2^2|^2$  does not change thus

$$\Delta(R, \tau_1, \tau_2, 1) \geq \frac{Vol(B(R^p))}{Vol(\tilde{\mathcal{F}})} = \frac{Vol(B(R^p))}{OT_3^2}. \quad (2.17)$$

**Remark 2.13** *Here we do not examine whether  $\mathcal{B}_\Gamma(R^p)$ ,  $\Gamma = \Gamma(\tau_1^p, \tau_2^p, 1)$  is a covering or not.*

We consider the Euclidean plane  $\alpha$  which passes through the point  $T_2^p$  and perpendicular to the axis  $y$ . If we move  $T_2^p$  in this plane then  $t_2^2$  is constant thus  $Vol(\tilde{\mathcal{F}}) = OT_3^2 = |t_1^1 \cdot t_2^2|^2$  does not change. By Theorem 2.10 and by Remark 2.11 we have obtained the following

**Lemma 2.14** *The radius  $R^p$  of the circumscribed ball of the Nil solid  $OT_1^pT_2^pT_3$  is minimal at the above moving of the point  $T_2^p$  if  $\alpha$  touch the Nil ball  $B(R^p)$  (see Fig. 5).*

**Remark 2.15** *At the "minimal position" of the point  $T_2^p$  is  $c_p^1 = t_2^1$ .*

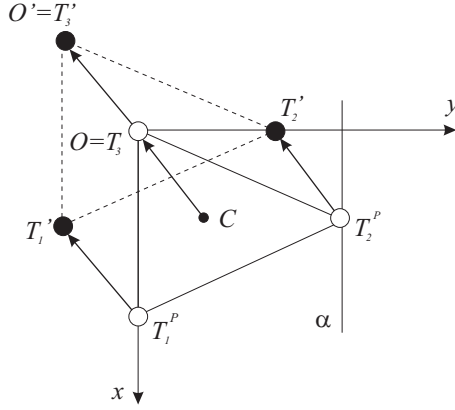


Figure 5:

Furthermore we shall decrease the radius of the circumscribed ball while the volume of the parallelepiped is constant. We translate the Nil solid  $OT_1^pT_2^pT_3$  and its circumscribed ball with Nil translation  $\overrightarrow{C^pO}$ , further we apply the quadratic mapping  $\mathcal{M} : \text{Nil} \rightarrow \mathbf{A}^3$  at (1.5) to this arrangement (see Fig. 5). At these transformations the volume of the parallelepiped  $\text{Vol}(\tilde{\mathcal{F}}) = OT_3^2$  and the volume of the circumscribed ball  $\text{Vol}(B(R^p))$  do not change. Thus we get a solid  $O'T_1'T_2'T_3'$  in affine space  $\mathbf{A}^3$  with its circumscribed ball of radius  $R^p$  centred by origin. This ball is convex in Euclidean sense (see Theorem 2.10). Moreover the points  $T_1', T_2'$  lie in the plane  $[x, y]$  and  $OT_3 = O'T_3' = \frac{1}{2}\mathcal{A}(H'T_1'T_2')$ . Here we have denoted by  $H' \in [x, y]$  the midpoint of the line segment  $O'T_3'$  and by  $\mathcal{A}(H'T_1'T_2')$  the area of the Euclidean triangle  $H'T_1'T_2'$ .

Working in analogy with what we know from Euclidean geometry, if we fix the volume of the parallelepiped  $\text{Vol}(\tilde{\mathcal{F}}) = OT_3 = O'T_3' = \frac{1}{2}\mathcal{A}(H'T_1'T_2')$  then by the Fig. 6. a,b,c,d can be derived the following

**Lemma 2.16** *The radius  $R^p$  of the circumscribed ball of the solid  $O'T_1'T_2'T_3'$  to a given volume of parallelepiped is minimal if  $H'T_1' = H'T_2'$  where  $H'T_1'$  and  $H'T_2'$  are Euclidean line segments.*

We have to examine these arrangements. To each  $R^p$  can be determined the line segment  $O'T_3'$  thus can be investigated the density function  $\Delta(R^p)$  of these orders and we need to test this function for a possible minimum if  $R^p \in [0, \frac{\pi}{2}]$ .  $\Delta(R^p)$  can be examined by careful computation with Maple. The graph of the function  $\Delta(R^p)$  can be seen in Fig. 7 and we get the following results:

$$\Delta(R_{min}^p) \approx 1.36278112, \quad R_{min}^p \approx 0.85847445. \quad (2.18)$$

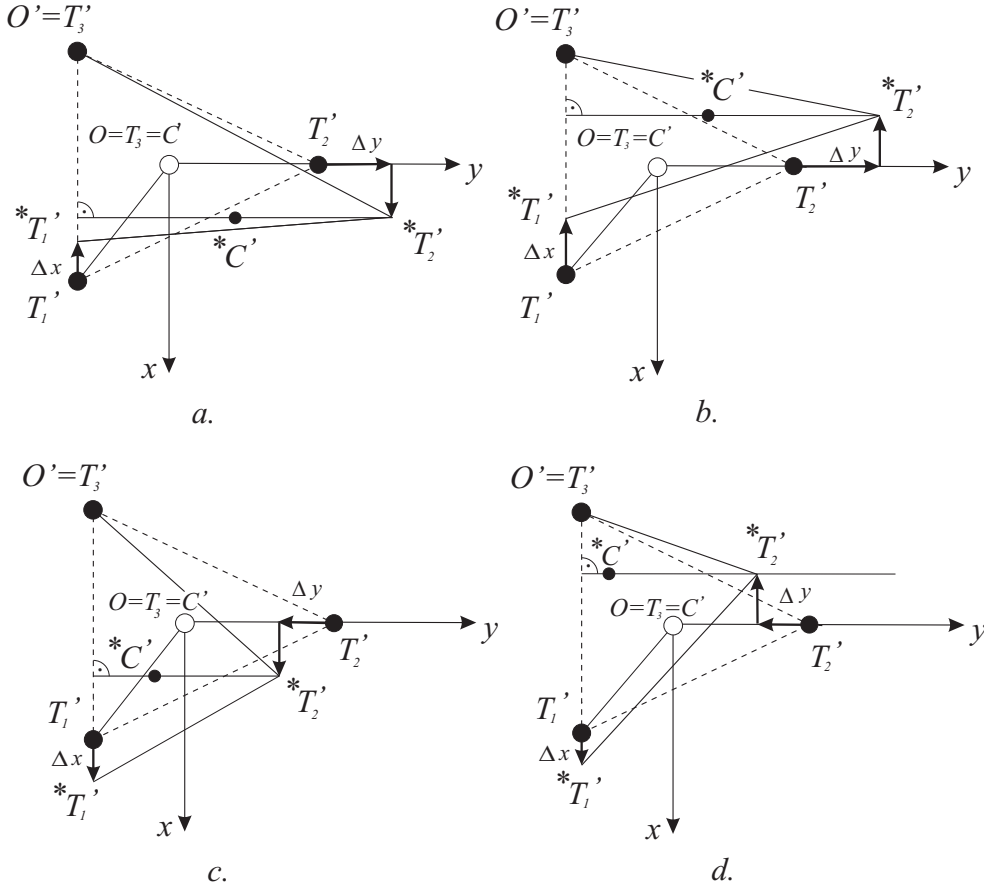


Figure 6:

Note that, it is easy to prove, that the ball arrangement belonging to the above given lattice does not yield a geodesic ball covering in the Nil space.

**Corollary 2.17** *Consequently, we have obtained a lower bound to the covering density:*

$$\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) > \Delta(R_{min}^p) \approx 1.36278112. \quad (2.19)$$

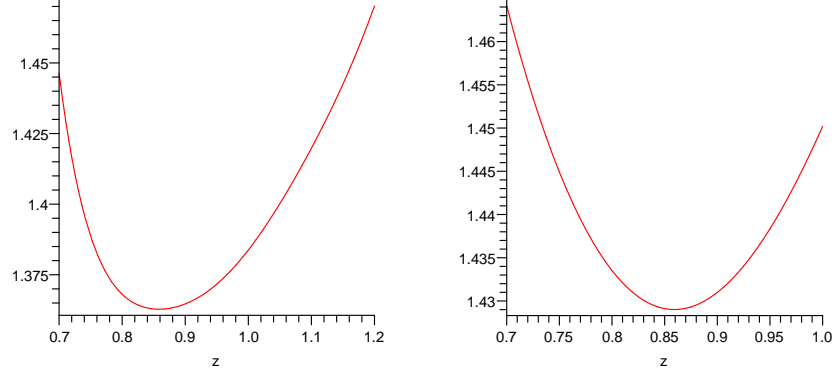


Figure 7: a, b

### 3 Conjecture for the least dense lattice-like ball covering in the Nil space

First we consider a Nil lattice  $\Gamma$  which is generated by the translations  $\mathbf{L}(\tau_1, \tau_2, 1)$  where  $\tau_1$  and  $\tau_2$  are given by the coordinates  $t_i^j$   $i = 1, 2; j = 1, 2, 3$  (see, Fig. 1). Moreover the points  $T_1$  and  $T_2$  lie on the equidistance surface of points  $O, T_3$  and  $t_2^1 = \frac{t_1^1}{2}$ ,  $t_2^2 = \frac{\sqrt{3}t_1^1}{2}$ , and one immediate consequence, that  $|t_3^3| = |t_1^1 \cdot t_2^1| = \sqrt{\text{Vol}(\mathcal{F})} = |(t_1^1)^2 \frac{\sqrt{3}}{2}|$ .

We have ball "columns" in  $z$ -direction and in regular hexagonal projection onto the  $[x, y]$  plane.

**Remark 3.1** *The lattice  $\Gamma_{opt}$  generated by the translations  $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt}, 1)$  (see (2.5)) is one of the above lattices.*

The radius  $R$  of the circumscribed ball of the Nil solid  $OT_1T_2T_3$  to a given parameter  $t_1^1$  can be determined by the following system of equation:

$$d(O, C) = d(C, T_3) = d(C, T_1) = d(C, T_2), \quad (3.1)$$

where  $C(1, c^1, c^2, c^3)$  is the center of the circumscribed ball of the point set  $\{0, T_1, T_2, T_3\}$ .

*In order to find the "suspected lower covering density" we investigate the*

density function

$$\Delta(R) = \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{\text{Vol}(\tilde{\mathcal{F}})} = \frac{\text{Vol}(\mathcal{B}_\Gamma(R))}{|(t_1^1)^2 \frac{\sqrt{3}}{2}|^2}. \quad (3.2)$$

To every  $R$  can be determined the parameter  $t_1^1$  thus can be examined the density function  $\Delta(R)$  of these arrangement and we need to test this function for a possible minimum if  $R \in [0, \frac{\pi}{2}]$ .  $\Delta(R)$  can be investigated by careful computation with Maple. The graph of the function  $\Delta(R)$  can be seen in Fig. 7.b and we get the following results:

$$\Delta(R_{min}^c) \approx 1.42900615, \quad R_{min}^c \approx 0.86046718, \quad (3.3)$$

$$\begin{aligned} t_1^{1,min} &\approx 1.26001585, \quad t_1^{3,min} \approx 0.68746826; \\ t_2^{1,min} &\approx 0,63000792, \quad t_2^{2,min} \approx 1,09120574, \quad t_2^{3,min} \approx 1.03120239, \\ T_1^{min} &= (1, t_1^{1,min}, 0, t_1^{3,min}), \quad T_2 = (1, t_2^{1,min}, t_2^{2,min}, t_2^{3,min}). \end{aligned} \quad (3.4)$$

Similarly to the Section 2.2 it is easy to see, that the ball arrangement belonging to the above given lattice is a geodesic ball covering in the Nil space, thus we get the following

### Theorem 3.2

$$1.36278112 \approx \Delta(R_{min}^p) < \Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) \leq \Delta(R_{min}^c) \approx 1.42900615.$$

**Conjecture 3.3** *The least dense lattice-like ball covering in the Nil space is derived by the Nil lattice  $\Gamma_{min}$  which is generated by the translations  $\mathbf{L}_{min}(\tau_1^{min}, \tau_2^{min}, 1)$  where  $\tau_1^{min}$  and  $\tau_2^{min}$  are given by the coordinates  $t_i^{j,min}$   $i = 1, 2; j = 1, 2, 3$ . The minimal covering radius  $R_{opt}^c = R_{min}^c \approx 0.86046718$  and*

$$\Delta_{opt}(R_{opt}^c, \tau_1^c, \tau_2^c, 1) = \Delta(R_{min}^c) \approx 1.42900615.$$

Our projective method gives us a way of investigation the Nil space, which suits to study and solve similar problems (see [6]). In this paper we have examined only some problems, but analogous questions in Nil geometry or, in general, in other homogeneous Thurston geometries are timely (see [7], [8], [9], [10]).

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